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q-functional Wick's theorems for particles with exotic statistics

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Abstract. In this paper we begin by giving a description of functional methods of quantum field theory for systems of interacting q-particles. These particles obey exotic statistics and are the q-generalization of the coloured particles which appear in many problems of condensed matter physics, magnetism and quantum optics. Motivated by the general ideas of standard field theory we prove the q-functional analogues of Hori's formulation of Wick's theorems for the different ordered q-particle creation and annihilation operators. *The formulae have the same formal expressions as fermionic and bosonic ones but differ by the nature of fields.* This allows us to derive the perturbation series for the theory and develop analogues of standard quantum field theory constructions in q-functional form.

1. Introduction

During the last decade many mathematical structures have been deformed and have gained the subscript q in their notations. In this way remarkable mathematical objects such as non-commutative geometries [1], quantum groups (q-groups) [2] and their representations on quantum vector spaces [3] have arisen. These objects led to the investigations of the quantum group gauge theory [4], q-deformed Schrödinger equation [5] and classical and quantum dynamics on q-deformed phase spaces [6]. In [7, 8] q-deformed spaces were considered as graded-commutative algebras. On this basis the classical and quantum dynamics on q-deformed spaces were proposed in close analogy with the case of $\mathbb{Z}_2$-commutative spaces (Grassmann algebras) [8].

After quantization the particles satisfy the q-deformed commutation relations similar to the commutation relations of the coloured particles [9]:

$$\begin{aligned} &a_k a_j \mp q a_j a_k = 0 \qquad a_k a_j^+ \mp q^{-1} a_j^+ a_k = 0 \qquad q = \mathrm{e}^{\mathrm{i}\alpha} \qquad 1 \leqslant k < j \leqslant n \\ &a_k a_k^+ \mp a_k^+ a_k = 1 \qquad \text{and for } q\text{-deformed fermions} \qquad [a_k^+]^2 = [a_k]^2 = 0 \end{aligned} \tag{1}$$

where the upper sign corresponds to quantization on q-deformed common space and the lower one corresponds to quantization on q-deformed Grassmann algebra. These particles

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were called q-particles to distinguish them from the famous q-oscillators which a number of papers are devoted to. These q-particles are the only objects that we consider.

It is surprising that q-particles are interesting, not only from the mathematical point of view but also from the physical one. Here we do not mean the appearance of q-particles in the parastatistics [7, 8, 10], q-extended supersymmetry [11], parasupersymmetry [10, 11] and other similar problems because they are rather mathematical applications. By physical applications we mean, first, solid state physics and quantum optics where q-particles appear in a natural way as well as the theory of magnetics where Paulions originally appeared and play a central role.

In solid state physics, anyons (particles with exotic braiding statistics) are important in some attempts to understand the physical features of planar systems [13]. The main physical interest in anyon systems is their possible connection with some effects in two-dimensional condensed matter physics, in particular in the quantum Hall effect [14] and high temperature superconductivity [15] (in the framework of the investigations of the t-J and Hubbard models).

In contrast to the previous example where anyons serve as auxiliary objects for the construction of one of possible scenarios there is a wide field in the quantum nonlinear optics in which q-particles are the main components. This is a theory of the collective behaviour of excitons with small radius (Frenkel excitons and charge-transfer excitons (CTE)) [16]. The studies investigate possibilities of formation of the Frenkel biexcitons and the observation of phase transitions in exciton systems in molecular crystals (Bose–Einstein condensation of excitons [17], structural phase transitions in crystals with high excitonic concentrations, dielectric-metal phase transition in a system of interacting CTE [18] and others). Strictly speaking excitons are not particles, they are quasiparticles describing molecular excitations and are of great importance in the analysis of nonlinear optical processes which accompany propagation of high-intensity light fluxes whose frequencies are in the range of the exciton absorption bands [19]. Moreover, excitons obey exotic statistics (Pauli statistics) [20] coinciding with q-particles statistics for $q = -1$. The general case of $q = \mathrm{e}^{\mathrm{i}\alpha}$ arises if we try to take into account phenomenologically some nonlinear effects (such as the difference in the creation time of molecular excitations for different types of molecules). This effect can be modelled by changing the Paulion commutation relations to those of the q-particles using the method developed in [21].

Surprisingly, even the investigation of the behaviour of low-dimensional exciton systems is meaningful. For example the exact solutions for one-dimensional Paulion chains [22] caused great advances in the theory of the so-called J-aggregates, i.e. molecular aggregates with unusually sharp absorption bands ([23] and references therein). The investigations of exciton systems on interfaces closely connect with the successes of contemporary technology. All these show that q-particles find deep applications in modern physical theories and motivate our objective to derive the appropriate field theoretical technique for them. The technique is developed in [24].

This paper is organized as follows. In section 2 we motivate the necessity for the introduction of q-ordered objects (such as the normal product) and introduce useful notation. In sections 3 and 4 we prove Wick's theorems in q-functional form for the simple product and q-symmetrical products of q-operators (q-symmetrical and q-chronological products). This allows us to find, in section 5, the Wick's theorems for the q-operator functionals. Section 6 contains the conclusions and some remarks.

2. Motivation and notations

In this section we motivate the necessity to introduce q-objects (such as N-, T-products of q-operators and others) which we study in the following sections. We also define 'universal' notations following [25] that allow us to prove Wick's theorems for the cases of q-operators with internal degrees of freedom (spin, colours) in the unified way.

As in the usual Fermi–Bose situation, in q-particle physics we are interested in two kinds of physical quantities. The first one is, generally speaking, the class of equilibrium thermodynamical characteristics calculated via the partition function of the system. The second one is a set of correlators which are important when describing the kinetics.

For clarity we consider a system of q-particles with creation (annihilation) operators obeying the commutation relations (1). The equilibrium thermodynamics of the model is described by the statistical operator $\rho = \mathrm{e}^{-\beta H}$ and the partition function $Z = \mathrm{Tr}\,\rho$. In close analogy with undeformed many-particle quantum theory there are several approaches for calculating Z. The comparative convenience of each of them is defined by the specific features of the system in question. The most straightforward way is to present ρ by the series

$$\rho = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} H^n$$

and to find matrix elements of the operators H^n in the occupation number basis $\{|\varphi_{i_1,i_2,\ldots,i_m}\rangle\} = \{a^+_{i_1} a^+_{i_2} \ldots a^+_{i_m}|0\rangle\}$ and then to try to calculate the sum of all diagonal matrix elements in this basis. Usually it is difficult enough to calculate the sum in a closed form.

To escape this problem one can introduce the basis coherent states corresponding to creation (annihilation) operators (1) [26]. These states use elements of the graded-commutative algebra $HQ^{0|L}$ (or $HQ^{L|0}$ in the bosonic case) as parameters. By proceeding in this way we need to calculate matrix elements of the operator in the Bargmann–Fock representation. This means that we need to find a representation of the operator in N-ordering form in which all creation operators are on the right-hand side with respect to all annihilation operators in all monomial terms. This can be done by using the Wick's theorem for q-operators in the same way as it was done for the usual Fermi–Bose statistics [25]. The next section is devoted to the formulation and proof of the theorem.

It is a very natural idea to use the objects of the non-commutative algebra for the investigations of systems with quantum particles obeying some exotic statistics. The first example of its useful application is in fermionic physics. Moreover, all functional methods of quantum bosonic field theory can be modified for the treatment of quantum fermionic fields using Grassmann variables instead of the usual complex numbers. This allows us to hope that the methods can be adopted for particles with exotic statistics (q-statistics) if permutation relations for the classical analogues are defined according to the rules of the statistics. In this way we can obtain a self-consistent scenario for building the quantum field theory (QFT) of particles with exotic statistics.

In this paper we consider creation and annihilation operators obeying the following commutation relations which generalize (1):

$$\left.\begin{array}{l}
\hat{a}(\boldsymbol{x})\hat{a}(\boldsymbol{y}) - \kappa q(\boldsymbol{x},\boldsymbol{y})\hat{a}(\boldsymbol{y})\hat{a}(\boldsymbol{x}) = 0 \\
\hat{a}^\dagger(\boldsymbol{x})\hat{a}^\dagger(\boldsymbol{y}) - \kappa q(\boldsymbol{x},\boldsymbol{y})\hat{a}^\dagger(\boldsymbol{y})\hat{a}^\dagger(\boldsymbol{x}) = 0 \\
\hat{a}(\boldsymbol{x})\hat{a}^\dagger(\boldsymbol{y}) - \kappa q^*(\boldsymbol{x},\boldsymbol{y})\hat{a}^\dagger(\boldsymbol{y})\hat{a}(\boldsymbol{x}) = \delta(\boldsymbol{x},\boldsymbol{y}) \\
[\hat{a}(\boldsymbol{x})]^2 = [\hat{a}^\dagger(\boldsymbol{x})]^2 = 0 \qquad \text{for } q\text{-fermions.}
\end{array}\right\} \tag{2}$$

This form allows us to consider continuous indices as well as discrete ones. $\boldsymbol{x}$ and $\boldsymbol{y}$ are in general D-dimensional vectors (or/and D-dimensional multi-indices for a lattice) which describe external and internal degrees of freedom. The function $\delta(\boldsymbol{x}, \boldsymbol{y})$ is a δ-function $\delta(\boldsymbol{x} - \boldsymbol{y})$ for a continuous space and Kronecker δ-function for a lattice or internal colour indices. The statistical factor $q(\boldsymbol{x}, \boldsymbol{y})$ possesses the following generalized Paulionic and anyonic property:

$$q(\boldsymbol{x}, \boldsymbol{y}) = q^{-1}(\boldsymbol{y}, \boldsymbol{x}) = q^*(\boldsymbol{y}, \boldsymbol{x}) \qquad q(\boldsymbol{x}, \boldsymbol{x}) = 1. \tag{3}$$

Finally κ serves to unify formulae for deformed bosonic and deformed fermionic cases. As usual it has the form:

$$\kappa = \begin{cases} +1 & \text{for } q\text{-bosons} \\ -1 & \text{for } q\text{-fermions.} \end{cases} \tag{4}$$

Hereafter letters with hats denote operators and those without hats denote the corresponding classical variables. For operator algebra (2) the corresponding classical variables satisfy the following permutation relations:

$$\left.\begin{aligned} & a(\boldsymbol{x})a(\boldsymbol{y}) - \kappa q(\boldsymbol{x}, \boldsymbol{y})a(\boldsymbol{y})a(\boldsymbol{x}) = 0 \\ & a^\dagger(\boldsymbol{x})a^\dagger(\boldsymbol{y}) - \kappa q(\boldsymbol{x}, \boldsymbol{y})a^\dagger(\boldsymbol{y})a^\dagger(\boldsymbol{x}) = 0 \\ & a(\boldsymbol{x})a^\dagger(\boldsymbol{y}) - \kappa q^*(\boldsymbol{x}, \boldsymbol{y})a^\dagger(\boldsymbol{y})a(\boldsymbol{x}) = 0 \\ & [a(\boldsymbol{x})]^2 = [a^\dagger(\boldsymbol{x})]^2 = 0 \qquad \text{for } q\text{-fermions.} \end{aligned}\right\} \tag{5}$$

Let us introduce 'universal' notations to avoid repetitions and enable us to formulate statements in unified form for different cases. Following [25] we collect the creation and annihilation operators into the single vector 'field' operator:

$$\hat{\varphi}(x) = \begin{pmatrix} \hat{a}^\dagger(\boldsymbol{x}) \\ \hat{a}(\boldsymbol{x}) \end{pmatrix} \qquad x \equiv (s, \boldsymbol{x}) \tag{6}$$

such that the additional vector index s indicates a type of operator ($\hat{a}^\dagger$ (when $s = 1$) or $\hat{a}$ (when $s = 2$)). It is convenient to represent quantities by vectors or matrices with respect to the index s.

Using this notation the commutation relations (2) can be rewritten in the following form:

$$\left.\begin{aligned} & \hat{\varphi}(x)\hat{\varphi}(y) - \kappa Q(x, y)\hat{\varphi}(y)\hat{\varphi}(x) = u(x, y) \\ & [\hat{\varphi}(x)]^2 = 0 \qquad \text{for } q\text{-fermions} \end{aligned}\right\} \tag{7}$$

where the statistical matrix $Q(x, y)$ is defined by statistical factor $q(\boldsymbol{x}, \boldsymbol{y})$:

$$Q(x, y) = \begin{pmatrix} q(\boldsymbol{x}, \boldsymbol{y}) & q^*(\boldsymbol{x}, \boldsymbol{y}) \\ q^*(\boldsymbol{x}, \boldsymbol{y}) & q(\boldsymbol{x}, \boldsymbol{y}) \end{pmatrix} \tag{8a}$$

and the quantum deformation parameter matrix is given by the equality:

$$u(x, y) = \delta(\boldsymbol{x}, \boldsymbol{y}) \begin{pmatrix} 0 & -\kappa \\ 1 & 0 \end{pmatrix}. \tag{8b}$$

The row number of the matrices from (8*a*) and (8*b*) corresponds to s_x and the column number to s_y. We can now associate classical analogues—classical vector 'field' variables—with the corresponding vector 'field' operator. For the classical variables we get the permutation relations in the following 'universal' form:

$$\left.\begin{aligned} & \varphi(x)\varphi(y) - \kappa Q(x, y)\varphi(y)\varphi(x) = 0 \\ & [\varphi(x)]^2 = 0 \qquad \text{for } q\text{-fermions} \end{aligned}\right\} \tag{9}$$

which is defined by the statistical matrix $Q(x, y)$ from (8*a*).

To formulate Wick's theorems in the functional form we need to introduce left $(\frac{\vec{\partial}}{\partial\varphi(x)})$ and right $(\frac{\overleftarrow{\partial}}{\partial\varphi(x)})$ functional derivatives on the algebra (9). We do as was done for the Grassmann algebra case: to act by the left (right) derivative on an expression we must bring the corresponding variable to the extremely left (right) position in accordance with the permutation relations and cancel it. This leads to the following commutation relations for the left derivative:

$$\frac{\vec{\partial}}{\partial\varphi(x)}\varphi(y) - \kappa Q^*(x, y)\varphi(y)\frac{\vec{\partial}}{\partial\varphi(x)} = \delta(x, y) \tag{10}$$

where $\delta(x, y) = \delta_{s_x, s_y}\delta(\boldsymbol{x}, \boldsymbol{y})$.

To construct Hori's functional expressions we will also need extended algebras which will contain several copies of the algebra of our vector 'field' variables as subalgebras. We define mutual permutation relations for the subalgebras to keep the graded-commutative structure. For example, the pair of fields $\varphi_1(x), \varphi_2(x)$ like (9) from different subalgebras commute with one another as follows

$$\varphi_1(x)\varphi_2(y) - \kappa Q(x, y)\varphi_2(y)\varphi_1(x) = 0. \tag{11}$$

We are now ready to proceed with Wick's theorems for operators (7).

3. Wick's theorem for the simple product of *q*-operators

In this section we formulate and prove the analogue of Hori's formula [27] which gives the functional form of the Wick's theorem for the normal form of a simple product of the creation and annihilation (or vector 'field') operators. *The formal expression does not depend on the statistics of the fields.* It shows that the exotic statistics can be taken into account by changing the permutation relations for the classical analogues according to the rules of the statistics and keeping the formal expression for Hori's formulae.

As was said above, we define the normal form $\mathrm{N}(A)$ of the monomial operator A as an expression where all creation operators are placed on the left-hand side in respect to the annihilation operators (using permutation relations (5)). Then the definition is generalized for an arbitrary polynomial operator by linearity. The definition is standard [25].

As usual we can do this in the vector 'field' form (in the 'universal' notations) where the final expressions only contain the normal contractions $n(x, y)$. For the pair of 'field' operators the normal product is defined by the following relation:

$$\mathrm{N}[\hat{\varphi}(x)\hat{\varphi}(y)] = \hat{\varphi}(x)\hat{\varphi}(y) - n(x, y). \tag{12}$$

For systems like (2), the normal contraction hence has the form:

$$n(x, y) = \delta(\boldsymbol{x}, \boldsymbol{y})\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

We note that the normal contraction has a δ-functional (or δ-symbol) character with respect to the indices. In the following we will frequently use this fact. More precisely we will use the fact that the product $\varphi(x)\varphi(y)$ is a C-number function when $n(x, y) \neq 0$.

We now prove the following theorem which reduces the simple operator product to the normal form.

Theorem 1.

$$\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n) = \mathrm{N}\left[\prod_{i<k}\left(1+\frac{\overleftarrow{\partial}}{\partial\varphi_i}n\frac{\overleftarrow{\partial}}{\partial\varphi_k}\right)\varphi_1(x_1)\ldots\varphi_n(x_n)\bigg|_{\ldots}\right] \tag{14}$$

where:

(1) the variables $\varphi_1,\ldots,\varphi_n$ form the extended algebra (9) following equation (11);

(2) the symbol $|_{\ldots}$ means substitution $\varphi_1=\cdots=\varphi_n=\hat{\varphi}$;

(3) substituting the differential forms into equation (14) in expanded form we have

$$\frac{\partial}{\partial\varphi_i}n\frac{\partial}{\partial\varphi_k} = \sum_{x,x'}\frac{\partial}{\partial\varphi_i(x)}n(x,x')\frac{\partial}{\partial\varphi_k(x')} \tag{15}$$

(here the symbol $\sum_x$ means summation over discrete variables and integration over continuous ones).

Proof. The strategy of the proof coincides with that of the proof of the Fermi–Bose Hori's formulae [25] and is fulfilled by induction. Indeed, for the particular cases $n=1,2$ the statement can be proved immediately:

$$\left.\begin{aligned}&\hat{\varphi}(x)=\mathrm{N}[\hat{\varphi}(x)]\\&\hat{\varphi}(x_1)\hat{\varphi}(x_2)=\mathrm{N}[\hat{\varphi}(x_1)\hat{\varphi}(x_2)+n(x_1,x_2)]\end{aligned}\right\} \tag{16}$$

by the application of the definitions of the normal form and the normal contraction (12), (13).

Let us now assume that equation (14) is true for any $n\leqslant N$ and consider the product of $N+1$ operators $\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_{N+1})$.

(1) If the field $\hat{\varphi}(x_{N+1})$ contains only the second component $\hat{a}(\boldsymbol{x}_{N+1})$ then

$$\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_{N+1}) = \hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N)\hat{a}(\boldsymbol{x}_{N+1}). \tag{17}$$

By the inductive assumption the first N multipliers on the right-hand side of equation (17) can be mapped to the normal form by the following reduction operator $\overleftarrow{\mathcal{P}}_N$:

$$\overleftarrow{\mathcal{P}}_N = \prod_{1\leqslant i<k\leqslant N}\left(1+\frac{\overleftarrow{\partial}}{\partial\varphi_i}n\frac{\overleftarrow{\partial}}{\partial\varphi_k}\right). \tag{18}$$

Then the full expression (17) also takes the normal form. It is not difficult to see that due to the structure of normal contraction (13) the differential expression (15) in (14) does not contain the derivative on $\hat{\varphi}^{(2)}(x_{N+1})=\hat{a}(\boldsymbol{x}_{N+1})$. Hence the terms with derivative $\frac{\overleftarrow{\partial}}{\partial\varphi_{N+1}}$ do not contribute in this case. So we can substitute $\overleftarrow{\mathcal{P}}_N$ by $\overleftarrow{\mathcal{P}}_{N+1}$ which completes the proof for this case.

(2) Let us now consider the case of $\hat{\varphi}(x_{N+1}) = \hat{a}^\dagger(\boldsymbol{x}_{N+1})$. Using N times the commutation relations (7) we obtain the following relation:

$$\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_{N+1}) = Q\hat{a}^\dagger(\boldsymbol{x}_{N+1})\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N)+\sum_{k=1}^{N}Q_k n(x_k,x_{N+1})[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N)]_k \tag{19}$$

where $[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N)]_k$ means product $\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N)$ without multiplier $\hat{\varphi}(x_k)$ and the following notations

$$\begin{aligned}&Q=\kappa^N Q(x_1,x_{N+1})\ldots Q(x_N,x_{N+1})\\&Q_k=\kappa^{N-k}Q(x_{k+1},x_{N+1})\ldots Q(x_N,x_{N+1})\end{aligned}$$

are introduced. The multipliers Q and Q_k reflect the q-deformation of the statistics and are due to the commutation relations (7). The first summand on the right-hand side of equation (19) can be rewritten as

$$\mathrm{N}[Q\hat{a}^\dagger(\boldsymbol{x}_{N+1})\cdot\overleftarrow{\mathcal{P}}_N\varphi_1\ldots\varphi_N|_{\ldots}]. \tag{20}$$

Note that under the sign of the normal products (20) the operator $\hat{a}^\dagger(\boldsymbol{x}_{N+1})$ can be substituted by the corresponding classical variable. Then if we move the variable $a^\dagger(\boldsymbol{x}_{N+1})$ to the original (right) position into expression (20) we gain the statistical multiplier Q^{-1} which exactly cancels the multiplier Q. This is because the reduction operator $\overleftarrow{\mathcal{P}}_N$ commutes with $a^\dagger(\boldsymbol{x}_{N+1})$ due to the δ-character of the normal contraction and the permutation relations for the classical analogues give the same statistical phases as the quantum ones. So

$$Q\hat{a}^\dagger(\boldsymbol{x}_{N+1})\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_N) = \mathrm{N}[\overleftarrow{\mathcal{P}}_N\varphi_1\ldots\varphi_{N+1}|_{\ldots}]. \tag{21}$$

Now consider the second summand on the right-hand side of (19). By applying the inductive assumption and substituting normal contraction by the expression in parentheses we get:

$$\mathrm{N}\left[\sum_{k=1}^{N} Q_k\left(\frac{\overleftarrow{\partial}}{\partial\varphi_k}n\frac{\overleftarrow{\partial}}{\partial\varphi_{N+1}}\varphi_k\varphi_{N+1}\right)\overleftarrow{\mathcal{P}}_N[\varphi_1\ldots\varphi_N]_k\Big|_{\ldots}\right].$$

Product $[\varphi_1\ldots\varphi_N]_k$ does not contain fields φ_k, φ_{N+1}. Using arguments similar to those for the derivation of (21) we can move them to thier original position and get

$$\mathrm{N}\left[\sum_{k=1}^{N}\frac{\overleftarrow{\partial}}{\partial\varphi_k}n\frac{\overleftarrow{\partial}}{\partial\varphi_{N+1}}\overleftarrow{\mathcal{P}}_N\varphi_1\ldots\varphi_{N+1}\Big|_{\ldots}\right]. \tag{22}$$

Collecting together equations (19), (21) and (22) we get

$$\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_{N+1}) = \mathrm{N}[\overleftarrow{\mathcal{P}}_{N+1}\varphi_1\ldots\varphi_{N+1}|_{\ldots}]. \tag{23}$$

Hence the proof of theorem 1 is complete. □

Corollary 1.1. Due to the linearity of expression (14) on each variable φ_i formulae (14) can be rewritten in the following form:

$$\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n) = \mathrm{N}\left[\exp\left[\sum_{i<k}\frac{\overleftarrow{\partial}}{\partial\varphi_i}n\frac{\overleftarrow{\partial}}{\partial\varphi_k}\right]\varphi_1(x_1)\ldots\varphi_n(x_n)\Big|_{\ldots}\right]. \tag{24}$$

Below we will use this relation to find the compact functional form for the Wick's theorems for the symmetrical and chronological products.

Corollary 1.2. Formulae (14), (24) hold true if we substitute right derivatives by left ones, sign '$<$' in product limits by '$>$' and normal contraction n by n^{T}.

Note here that the formal form of the statement of theorem 1 does not depend on any statistics (i.e. it is the same as in fermionic and bosonic cases) and the commutation factors are 'hidden' in the nature of the classical variables. It is useful to remember this when studying all other theorems of the paper where it is also true.

4. Wick's theorem for q-symmetrical products

In this section we consider other kinds of products, in particular, q-symmetrical and q-chronological. Such products naturally play the role of the usual symmetrical for bosons (antisymmetrical for fermions) and chronological products in the framework of the standard quantum field theory. Moreover, they are interesting by themselves because they keep the symmetry properties of the q-operators.

In close analogy with the undeformed cases we define q-symmetrical product (Sym_q-product) of q-operators in the following way

$$\mathrm{Sym}_q[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] = \frac{1}{n!}\sum_P Q_P P[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)]. \tag{25}$$

The sum is taken over all $n!$ permutations P of the q-operators $\hat{\varphi}$ with corresponding statistical phases Q_P. These Q_P are the factors arising from permutation of operators from the original order to the order P. We assume that under the permutation q-operators $\hat{\varphi}$ are replaced by the corresponding classical variables $\hat{\varphi} \to \varphi$ and we do not pick up the expressions due to the right-hand side of equation (7). In other words, Q_P is defined from the relation

$$\varphi(x_1)\ldots\varphi_n(x_n) = Q_P P[\varphi(x_1)\ldots\varphi_n(x_n)]. \tag{26}$$

We now consider the q-chronological product. We assume that q-operators $\hat{\varphi}$ also depend on time t and define q-chronological product (or T_q-product) of the q-operators by the equation (all times t_i are assumed to be different):

$$\mathrm{T}_q[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] = \sum_P Q_P P[\theta(1\ldots n)\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] \tag{27}$$

where the n-point θ-function is given by the relation:

$$\theta(1\ldots n) \equiv \prod_{k=1}^{n-1}\theta(t_k - t_{k+1}). \tag{28}$$

The summation in equation (27) is taken over all simultaneous permutations of the operators $\hat{\varphi}(x_i)$ and the corresponding times t_i in θ-function.

If all or any part of the arguments t_i of the operators are equal then the T_q-product is not rigorously defined. We need to complete the definition. We define the T_q-product under equal times as the Sym_q-product. An action of Sym_q- and T_q-products on zeroth power of operators is defined as usual by the equalities: $\mathrm{Sym}_q[1] = \mathrm{T}_q[1] = 1$.

We would like to note that Sym_q and T_q (similar to N) are not true linear operators on the space of the field operators: an operator equality $\hat{F}_1 = \hat{F}_2$ is not followed by the equalities $\mathrm{T}_q\hat{F}_1 = \mathrm{T}_q\hat{F}_2$ or $\mathrm{Sym}_q\,\hat{F}_1 = \mathrm{Sym}_q\,\hat{F}_2$ (because the field operators in the argument of operations T_q, Sym_q, N behave themselves as the classical variables).

We call a product *q-symmetric* if under permutation multipliers (q-operators) in the product it behaves in the same manner as if the q-operators would be classical variables, i.e. only the corresponding statistical phase appears (26). This definition naturally generalizes the notion of (anti)symmetry of the field operator products in the Bose (Fermi) case. For example, the normal product is q-symmetric

$$\mathrm{N}[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] = Q_P\mathrm{N}[P\{\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)\}]. \tag{29}$$

It is also not difficult to check that the Sym_q- and T_q-products introduced above are q-symmetric:

$$\begin{aligned}\mathrm{Sym}_q[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] &= Q_P\,\mathrm{Sym}_q[P\{\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)\}]\\ \mathrm{T}_q[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] &= Q_P\,\mathrm{T}_q[P\{\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)\}].\end{aligned} \tag{30}$$

To formulate the functional form of the Wick's theorem for the above-defined products we need to introduce the q-chronological contraction $\Delta_q(x, y)$ and the q-symmetrical contraction (or the q-symmetric part of the normal contraction n) n_s^q. They are defined by the relations

$$\mathrm{T}_q[\hat{\varphi}(x)\hat{\varphi}(y)] = \mathrm{N}[\hat{\varphi}(x)\hat{\varphi}(y)] + \Delta_q(x, y) \tag{31}$$

$$\mathrm{Sym}_q[\hat{\varphi}(x_1)\hat{\varphi}(x_2)] = \mathrm{N}[\hat{\varphi}(x_1)\hat{\varphi}(x_2)] + n_s^q(x_1, x_2). \tag{32}$$

Using definitions (12), (25), (27) we get expressions for the contraction via the normal one (13):

$$\Delta_q = \theta(12)n + \kappa Q\theta(21)n^{\mathrm{T}} = \theta(12)n(x_1, x_2) + \kappa Q(x_1, x_2)\theta(21)n(x_2, x_1) \tag{33}$$

$$n_s^q = \tfrac{1}{2}[n + \kappa Q n^{\mathrm{T}}] = \tfrac{1}{2}[n(x_1, x_2) + \kappa Q(x_1, x_2)n(x_2, x_1)]. \tag{34}$$

Due to our definition of the q-chronological product on equal times we have

$$\Delta_q(x_1, x_2)|_{t_1=t_2} = n_s^q(x_1, x_2)|_{t_1=t_2}. \tag{35}$$

We wish to note that the q-chronological contraction (similar to n_s^q), in contrast to the normal contraction n, has the property of q-symmetry, i.e. $\Delta_q = \kappa Q \Delta_q^{\mathrm{T}}$.

We are now ready to formulate the Wick's theorem for q-symmetric products which gives the Sym_q- and T_q-products in the normal form.

Theorem 2.

$$\mathrm{Sym}_q[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] = \mathrm{N}\left[\exp\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\varphi}n\frac{\overleftarrow{\partial}}{\partial\varphi}\right)\varphi(x_1)\ldots\varphi(x_n)\bigg|_{\varphi=\hat{\varphi}}\right] \tag{36}$$

$$\mathrm{T}_q[\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n)] = \mathrm{N}\left[\exp\left(\frac{\overleftarrow{\partial}}{\partial\varphi}\Delta_q\frac{\overleftarrow{\partial}}{\partial\varphi}\right)\varphi(x_1)\ldots\varphi(x_n)\bigg|_{\varphi=\hat{\varphi}}\right]. \tag{37}$$

The proof can be performed in close analogy with the undeformed case [25] and the proof of theorem 1.

Corollary 2.1. The normal contraction n in equation (36) can be replaced by n_s^q from (34) due to the fact that the kernel of the differential operation is automatically symmetrized:

$$\frac{\partial}{\partial\varphi_i}n\frac{\partial}{\partial\varphi_k} = \frac{\partial}{\partial\varphi_k}\kappa Q n^{\mathrm{T}}\frac{\partial}{\partial\varphi_i} \qquad \frac{\partial}{\partial\varphi}n\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial\varphi}n_s^q\frac{\partial}{\partial\varphi}.$$

5. Wick's theorems for q-operator functionals

To describe the Green function technique and perturbation theory for the systems of the q-deformed particles we also need the rules for finding the normal form not only of the products of the operators but the whole functionals as well. A q-operator expression $F(\hat{\varphi})$ is said to be *q-operator functional* if it has the following form:

$$F(\hat{\varphi}) = \sum_{n=0}^{\infty}\int\ldots\int \mathrm{d}x_1\ldots\mathrm{d}x_n\, F_n(x_1, \ldots, x_n)\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n). \tag{38}$$

The operator functional is completely defined by the set of its (may be generalized and singular) coefficient functions $F_n(x_1 \ldots x_n)$. We will call an operator functional *q-symmetrical* if all its coefficient functions satisfy the following relations:

$$\begin{aligned} &F_n(\ldots x_i \ldots x_k \ldots) = Q_{ik}^{-1} F_n(\ldots x_k \ldots x_i \ldots) \\ &\ldots \varphi(x_i) \ldots \varphi(x_k) \ldots = Q_{ik}(\ldots \varphi(x_k) \ldots \varphi(x_i) \ldots). \end{aligned} \tag{39}$$

Classical functionals (which are obtained from the operator one by replacing $\hat{\varphi}$ by φ) correspond to the q-symmetrical operator functional. q-symmetrical coefficient functions are uniquely determined by the classical functional. Note also that any q-symmetrical functional possesses the property $F(\hat{\varphi}) = \mathrm{Sym}_q\, F(\hat{\varphi})$. We call an operator functional $F(\hat{\varphi})$ as *an operator functional in normal form* if it possesses the property $F(\hat{\varphi}) = \mathrm{N}[F(\hat{\varphi})]$.

We emphasize that operator functionals are defined just by their coefficient functions (not by operator $F(\hat{\varphi})$). The functions F_n determine operator $F(\hat{\varphi})$ uniquely but the opposite statement is not true in general.

Formulae (36) and (37) are generalized directly to operator functionals due to the 'universality' of the reduction operation (we assume that operator functionals do not contain time derivatives, although all considerations can be extended to this case). Thus we obtain the following rules for reducing operator functionals to the normal form which we collate into theorem 3.

Theorem 3.

$$\mathrm{Sym}_q\, F(\hat{\varphi}) = \mathrm{N} \exp\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial\varphi} n \frac{\overleftarrow{\partial}}{\partial\varphi}\right) F(\varphi)\bigg|_{\varphi=\hat{\varphi}} \tag{40}$$

$$\mathrm{T}_q F(\hat{\varphi}) = \mathrm{N} \exp\left(\frac{\overleftarrow{\partial}}{\partial\varphi} \Delta_q \frac{\overleftarrow{\partial}}{\partial\varphi}\right) F(\varphi)\bigg|_{\varphi=\hat{\varphi}}. \tag{41}$$

Corollary 3.1. From these various formulae, the inverse and combined transformations can be easily found. For example

$$\mathrm{N} F(\hat{\varphi}) = \mathrm{Sym}_q \exp\left(-\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial\varphi} n \frac{\overleftarrow{\partial}}{\partial\varphi}\right) F(\varphi)\bigg|_{\varphi=\hat{\varphi}} \tag{42a}$$

$$\mathrm{T}_q F(\hat{\varphi}) = \mathrm{Sym}_q \exp\left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial\varphi} (\Delta_q - n) \frac{\overleftarrow{\partial}}{\partial\varphi}\right) F(\varphi)\bigg|_{\varphi=\hat{\varphi}}. \tag{42b}$$

We proceed with the consideration of the normal form of a product of the operator functionals. We formulate the Wick's theorem for the product of operator functionals *in normal form*.

Theorem 4.

$$\prod_{k=1}^{n} [F^{(k)}(\hat{\varphi})] = \mathrm{N}\left\{ \exp\left[\sum_{i<k} \frac{\overleftarrow{\partial}}{\partial\varphi_i} n \frac{\overleftarrow{\partial}}{\partial\varphi_k}\right] \prod_{k=1}^{n} F^{(k)}(\varphi_k)\bigg|_{\ldots} \right\}. \tag{43}$$

Hereafter the non-commuting multipliers are believed to be ordered in accordance with increasing index:

$$\prod_{k=1}^{n} [F^{(k)}(\hat{\varphi})] \equiv F^{(1)}(\hat{\varphi}), \ldots, F^{(n)}(\hat{\varphi}). \tag{44}$$

Corollary 4.1. Starting from the basic Wick's theorem (14) and arguing as when deriving equation (36) one can obtain the following rule of reduction of a product of *q-symmetrical functionals* to the normal form:

$$\prod_{k=1}^{n}[\mathrm{Sym}_q\, F^{(k)}(\hat{\varphi})] = \mathrm{N}\left\{ \exp\left[\frac{1}{2}\sum_i \frac{\overleftarrow{\partial}}{\partial\varphi_i} n \frac{\overleftarrow{\partial}}{\partial\varphi_i} + \sum_{i<k} \frac{\overleftarrow{\partial}}{\partial\varphi_i} n \frac{\overleftarrow{\partial}}{\partial\varphi_k}\right] \prod_{k=1}^{n} F^{(k)}(\varphi_k)\bigg|_{\dots} \right\}. \tag{45}$$

Corollary 4.2. Equation (45) is obviously generalized to the case when some multipliers on the left-hand side are given in the N-form (not Sym_q-form):

$$\prod_{k=1}^{n}[\mathcal{A}F^{(k)}(\hat{\varphi})] = \mathrm{N}\left\{ \exp\left[\frac{1}{2}\sum_{\mathrm{Sym}_q} \frac{\overleftarrow{\partial}}{\partial\varphi_i} n \frac{\overleftarrow{\partial}}{\partial\varphi_i} + \sum_{i<k} \frac{\overleftarrow{\partial}}{\partial\varphi_i} n \frac{\overleftarrow{\partial}}{\partial\varphi_k}\right] \prod_{k=1}^{n} F^{(k)}(\varphi_k)\bigg|_{\dots} \right\}. \tag{46}$$

Here $\mathcal{A}$ denotes Sym_q or N and summation in the diagonal terms of the quadratic form is only over functionals that stand in Sym_q-form.

If a product of operator functionals stands under the common sign of some q-symmetrical product (Sym_q or T_q) it is automatically q-symmetrized and one can use the usual formulae (40) and (41). For example,

$$\mathrm{T}_q\left[\prod_{k=1}^{n} F^{(k)}(\hat{\varphi})\right] = \mathrm{N}\exp\left(\frac{\overleftarrow{\partial}}{\partial\varphi}\Delta_q\frac{\overleftarrow{\partial}}{\partial\varphi}\right)\prod_{k=1}^{n} F^{(k)}(\varphi_k)\bigg|_{\varphi=\hat{\varphi}}. \tag{47}$$

In conclusion we note that it is possible to complete the definition of the T_q-product not via the Sym_q-product but by using the N-form. This is equivalent to the definition of the θ-function entering into the chronological contraction under coincided times as follows $\theta(t_{12})|_{t_1=t_2} = 0$. It is not difficult to check that formula (37) remains true under this convention. The same can be said about formula (41) but the classical functional entering into this formula should represent N-form of the corresponding q-operator functional, i.e. $F(\hat{\varphi}) = \mathrm{N}F(\hat{\varphi})$.

6. Conclusion

In this paper we have formulated Wick's theorems for general q-operators and q-operator functionals. The formal expression does not depend on the statistics of the fields and the statistical aspect is revealed only when the nature of the algebra of the corresponding classical variables is defined. This implies that the functional approach to the Wick's theorems using the analogue of the Hori's formulae is the most natural one. Moreover, the theorems of the paper show that it is possible to take into account the exotic statistics keeping the formal expression for the Hori's formulae and changing the permutation relations for the classical analogues according to the rules of the statistics. This fact will allow us to derive the perturbation series for the theory and develop analogous constructions of standard quantum field theory in q-functional form in a straightforward way (see [24] for details).

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References

[1] Connes A 1986 *Non-commutative Differential Geometry* Publ. Math. I.H.E.S. N62

[2] Faddeev L D, Reshetikhin N Yu and Takhtadzhyan L A 1987 *Algebra and Analysis* **1** 178
Majid S 1990 *Int. J. Mod. Phys.* A **5** 1

[3] Manin Yu I 1988 Quantum groups and non-commutative geometry *Preprint* Montreal University CRM-1561

[4] Aref'eva I Ya and Volovich I V 1991 *Mod. Phys. Lett.* A **6** 893
Isaev A P and Popowicz Z 1992 *Phys. Lett.* B **281** 271
Castellani L 1992 *Phys. Lett.* B **292** 93
Bernard D 1992 *Suppl. Prog. Theor. Phys.* **102** 49
Hirayama M 1992 *Prog. Theor. Phys.* **88** 111

[5] Wess J and Zumino B 1990 *Nucl. Phys. (Proc. Suppl.)* B **18** 302
Carow-Watamura U, Schlieker M and Watamura F 1991 *Z. Phys.* C **49** 439

[6] Aref'eva I Ya and Volovich I V 1991 *Phys. Lett.* B **268** 179
Rembielinski J and Smolinski K A 1993 Non-commutative quantum dynamics *Preprint* KFT UL 3/93
Ubriaco M R 1993 *Mod. Phys. Lett.* A **8** 89

[7] Marcinek W 1992 *J. Math. Phys.* **33** 1631

[8] Borisov N V, Ilinski K N and Uzdin V M 1992 *Phys. Lett.* A **169** 427

[9] Ohnuki Y and Kamefuchi S 1980 *J. Math. Phys.* **21** 601

[10] Filippov A T 1992 *Mod. Phys. Lett.* A **7** 2129

[11] Ilinski K N and Uzdin V M 1993 *Mod. Phys. Lett.* A **8** 2657

[12] Beckers J and Debergh N 1991 *J. Phys. A: Math. Gen.* **24** L597

[13] Fradkin E 1991 *Field Theories of Condensed Matter Systems* (Reading, MA: Addison-Wesley)

[14] Girvin S and MacDonald A 1987 *Phys. Rev. Lett.* **58** 1252

[15] Wilczek F (ed) 1990 *Fractional Statistics and Anyon Superconductivity* (Singapore: World Scientific)

[16] Agranovich V M, Galanin M D 1982 *Electronic Excitation Energy Transfer in Condensed Matter* (Amsterdam: North-Holland) p 381

[17] Agranovich V M and Toshich B S 1967 *Zh. Eksp. Teor Fiz.* **53** 149

[18] Agranovich V M and Ilinski K N 1994 *Phys. Lett.* A **191** 309

[19] Agranovich V M 1968 *Theory of Excitons* (Moscow: Nauka) p 384 (in Russian)

[20] Agranovich V M 1959 *Zh. Eksp. Teor Fiz.* **37** 430

[21] Ilinski K N and Uzdin V M 1993 *Phys. Lett.* A **174** 179–81

[22] Lieb E, Schults T and Mattis D 1961 *Ann. Phys., NY* **16** 407

[23] Spano F S and Knoester J 1994 *Adv. Magnet. Opt. Res.* **18** 117

[24] Ilinski K N, Kalinin G V and Stepanenko A S *Phys. Lett.* A to appear

[25] Vasiliev A N 1976 *Functional Methods in Quantum Field Theory and Statistics* (Leningrad: Leningrad University Press) (in Russian)

[26] Ilinski K N and Stepanenko A S 1994 *Phys. Lett.* A **187** 1

[27] Hori S 1952 *Prog. Theor. Phys.* **7** 578